Zero-dimensional spaces from linear structures.

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Abstract

It is well known that spaces defined from a separated uniform structure with a linearly ordered base of uncountable cofinality (ω_μ-metrizable spaces) are ultraparacompact and have an ortho-base, hence are non-Archimedean spaces in the sense of A. Monna. Our results concern whether certain wider classes of spaces defined from linear structures retain these properties. We construct for every regular cardinal ω_μ, examples of ω_μ-additive, ω_μ-stratifiable spaces (i.e., ω_μ-Nagata spaces) that do not have an ortho-base. We give a number of examples of linearly stratifiable spaces, one of which is related to an example of Eric K. van Douwen concerning countable box products of stratifiable spaces.

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1 Introduction

Zero-dimensional spaces arise throughout topology, in analysis, and also in several contexts in algebra such as the Stone spaces of Boolean algebras, and the $\mathfrak{m}$-adic topologies from local rings. The structure of zero-dimensional spaces from a "geometric" point of view was considered by P. Nyikos and H. C. Reichel [19], who also provided a review of zero-dimensional spaces.

This paper concerns the structure of and differences among several classes of zero-dimensional topological spaces defined from linear structures such as separated uniform structures with a linearly ordered base of cofinality ω_μ (or equivalently, ω_μ-metrics in the sense of F. Hausdorff [12, p.285], where ω_μ denotes a regular infinite cardinal), and the more general linearly stratifiable spaces.

A well-known class of zero-dimensional spaces is the class of non-Archimedean spaces in the sense of A. F. Monna [16]. These spaces have been characterized by P. Nyikos as spaces that are both ultraparacompact (see 2.4) and have an ortho-base (see 2.3) [18]. A. Kucia and W. Kulpa proved that for ω_μ uncountable (i.e., $\mu > 0$) every ω_μ-metric space is ultraparacompact [13], and P. Nyikos proved that ω_μ-metric spaces have an ortho-base [18] . Thus ω_μ-metric spaces ($\mu > 0$) are non-Archimedean.

The ω_μ-metric spaces have two important linear properties. One is the manner in which open sets in an ω_μ-metric space (X, d) are increasing unions of closures of open sets: for U open in X, and $\epsilon > 0$, U is the increasing union of the closures of the sets $\{y : d(y, X \setminus U) < \epsilon\}$ (linearly stratifiable spaces were defined to capture this property). The other (partially) linear property, which is of interest in the case $\mu > 0$, is the property that every intersection of countably many open sets is open (i.e., ω_1-additivity). Our main results show that ω_1-additive paracompact spaces, hence ω_1-additive ω_μ-stratifiable spaces, are ultraparacompact, but need not have an ortho-base (see 3.1, and 4.1). Thus such spaces, although defined using two important properties of ω_μ-metric spaces, constitute a distinctly different class of zero-dimensional spaces since they are not non-Archimedean spaces. An unsolved question is whether the structure of linear stratification alone (without ω_1-additivity) implies ultraparacompactness or zero-dimensionality.

Towards understanding the class of linearly stratifiable spaces, we give a number of examples of such spaces. These examples are related to results in dimension theory, the theory of ordered spaces, and set theory.

The countable cases (i.e., $\mu = 0$) correspond to ω_0-metric spaces (which are the same as the usual metric spaces) and spaces stratifiable over ω_0 (which are the same as the usual stratifiable spaces of J. Ceder [2] and C. Borges [1]). Several of our examples are of interest in the countable case. Our result in §4 provides the new result that there exists a Nagata space (i.e., first countable, stratifiable) that does not have an ortho-base. In §6, we give an example, related to one of Eric K. van Douwen, that shows the countable box product of stratifiable spaces can fail to be stratifiable, even if the product is ω_1-stratifiable, hence hereditarily paracompact and monotonically normal. We give two examples concerning pseudocharacter and character in ω_μ-stratifiable spaces (see §7).

2 Definitions

We introduced in 1972 higher cardinal versions of two generalized metric spaces, the Nagata and stratifiable spaces, which we called ω_μ-Nagata and stratifiable over ω_μ. We recall these definitions.

Definition 2.1 ([22]) *A space X with topology $\mathcal{T}$ is said to be* stratifiable over ω_μ *provided there exists $S : \omega_\mu \times \mathcal{T} \to \mathcal{T}$ satisfying for all $U \in \mathcal{T}$, and $\beta < \omega_\mu$*

LS_1 $cl_X[S(\beta, (U)] \subset U$.

LS_2 $\bigcup\{S(\beta, U) : \beta < \omega_\mu\} = U$.

LS_3 if $W \in \mathcal{T}$ and $U \subset W$, then $S(\beta, U) \subset S(\beta, W)$.

LS_4 if $\gamma < \beta < \omega_\mu$, then $S(\gamma, U) \subset S(\beta, U)$.

*The function S is called an ω_μ-*stratification map (or stratification map) for *X.*

Definition 2.2 *A point x in a space X is said to have a* decreasing local neighborhood base of cofinality ω_μ *provided for each $x \in X$ there exists a local base $\{N_\alpha(x) : \alpha < \omega_\mu\}$ of neighborhoods of x such that for every $\alpha < \beta < \omega_\mu$, $N_\beta(x) \subset N_\alpha(x)$. X is called* ω_μ-Nagata *provided X is stratifiable over ω_μ and every point has a decreasing local base indexed by ω_μ.*

The preceding definition of ω_μ-Nagata space is equivalent to the original one [22, Theorem 6.3].

If a space is stratifiable over at least one regular cardinal, we say it is *linearly stratifiable*, and if ω_μ is the smallest cardinal over which a space X is stratifiable, we say that X is *ω_μ-stratifiable*.

We now define three properties that play a major role in the study of linearly stratifiable spaces.

Definition 2.3 *[18] A base $\mathcal{B}$ is called an* ortho-base *for a space X provided for every $\mathcal{B}' \subset \mathcal{B}$ either $\cap\mathcal{B}'$ is open, or $\cap\mathcal{B}'$ consists of a single point x and $\mathcal{B}'$ is a local base for x.*

Definition 2.4 *[7, Proposition 1.2] A space X is called* ultraparacompact *provided every open cover of X has a refinement consisting of pairwise disjoint clopen sets (i.e., a refinement which is a partition of X into clopen sets).*

Recall that P. Roy [20] gave an example of a metric space Δ such that ind $(\Delta) = 0$ and Ind $(\Delta) = 1$. This shows that even a zero-dimensional metric space (i.e., $\mu = 0$) need not be ultraparacompact.

Definition 2.5 *[21] A space is called ω_μ-*additive *provided every intersection of fewer than ω_μ open sets is open.*

Obviously any space with a decreasing local base of cofinality ω_μ (ω_μ regular) at every point is ω_μ-additive; so ω_μ-Nagata spaces are ω_μ-additive. In contrast, a space stratifiable over ω_μ need not be ω_μ-additive (nor even ω_1-additive). This may be seen easily in the spaces constructed in §5.

We mention Nyikos's theorem that if a space X is stratifiable over ω_μ, and has an ortho-base, then X is ω_μ-metrizable [18].

3 Ultraparacompactness

In this section we show that every ω_1-additive linearly stratifiable space is ultraparacompact. Nyikos informed us that ω_1-additive paracompact spaces are ultraparacompact, and moreover this can be deduced from known results.

Theorem 3.1 *Every ω_1-additive paracompact space is ultraparacompact. In particular, every ω_1-additive linearly stratifiable space is ultraparacompact (thus ω_μ-Nagata space ultraparacompact when $\mu > 0$).*

Proof. Let X be paracompact and ω_1-additive. Sikorski [21] noted that every ω_1-additive normal space has the property that whenever $F \subset U$ with F closed and U open, there exists a clopen set $F \subset V \subset U$ (i.e., Ind $X = 0$, or X is ultranormal [6]). R. Ellis proved that paracompact ultranormal spaces are ultraparacompact [7, Proposition 1.2]. One could give an alternate proof using two results from the dimension theory of normal spaces (1) Ind $X = 0$ if and only if dim $X = 0$ [17, 8-3], and (2) dim $X = n$ if and only if every locally finite open cover has an open refinement of order $n+1$ (an open cover of order 1 is a pairwise disjoint open cover) [5]. This proves the first statement in Theorem 3.1. To complete the proof we call on the following result.

Theorem 3.2 *Every linearly stratifiable space is paracompact.*

We stated Theorem 3.2 in [22], and said it followed from our characterization of paracompactness in [23], but J. Harris pointed out to us that it is not clear how to directly apply the characterization. In her dissertation, Harris gave an alternate proof of Theorem 3.2 using a theorem of hers of independent interest [11]. It seems we should have said that Theorem 3.2 follows from the *idea of the proof* of our characterization in [23]. For completeness, we now give that proof incorporating a simplification suggested by the referee.

Proof of Theorem 3.2. By the Theorem of E. Michael [15], it suffices to prove that every open cover $\mathcal{U}$ of X has a cushioned refinement $\mathcal{V}$ (i.e, $\mathcal{V}$ is a (not necessarily open) refinement of $\mathcal{U}$ and there exists a function $f : \mathcal{V} \to \mathcal{U}$ such that for every $\mathcal{V}' \subset \mathcal{V}$, we have $cl_X(\cup\mathcal{V}') \subset \cup f(\mathcal{V}'))$. Well-order $\mathcal{U}$ and let $\lhd$ be the lexicographic order on $\omega_\mu \times \mathcal{U}$. Define

$$H(\alpha, U) = S(\alpha, U) \setminus \bigcup\{S(\beta, V) : (\beta, V) \lhd (\alpha, U)\}.$$

Every $x \in X$ is in some $S(\alpha, U)$, hence in $H(\alpha, U)$ for the first such (α, U). We show that the map $H(\alpha, U) \mapsto U$ is a cushion map. Let $A \subset \omega_\mu \times \mathcal{U}$, and $x \in cl_X(\bigcup\{H(\alpha, U) : (\alpha, U) \in A\})$. Fix α, U with $x \in S(\alpha, U)$, and note that $S(\alpha, U) \cap H(\beta, V) = \emptyset$ whenever $(\alpha, U) \lhd (\beta, V)$. Thus

$$x \in cl_X(\bigcup\{H(\beta, V) : (\beta, V) \in A \text{ and } (\beta, V) \lhd (\alpha, U)\}.$$

If $(\beta, V) \lhd (\alpha, U)$, then $\beta \leq \alpha$, and thus $H(\beta, V) \subset S(\beta, V) \subset S(\alpha, \bigcup\mathcal{U}_A)$, where $\bigcup\mathcal{U}_A = \{U \in \mathcal{U} : (\exists\alpha)((\alpha, U) \in A)\}$. It follows that $x \in cl_X(S(\alpha, \bigcup\mathcal{U}_A) \subset \bigcup\mathcal{U}_A$.

Sikorski also noted that if ω_μ is an uncountable regular cardinal then every regular, ω_μ-additive space is zero-dimensional (i.e., has a base of clopen sets) [21]. This leads to an open question.

Question 3.3 *If X is stratifiable over an uncountable regular cardinal ω_μ, is X zero-dimensional, or ultraparacompact?*

4 An ω_μ-Nagata space not having an ortho-base

The example in this section was inspired by the bow-tie space of Louis McAuley [14], which is a standard example of a Nagata space that is not metrizable.

We establish some notation. For an infinite cardinal ω_μ let $L(\omega_\mu)$ denote the space $\omega_\mu + 1$ with the topology in which all $\alpha < \omega_\mu$ are isolated, and the point ω_μ has its usual order neighborhoods. For every $\alpha \leq \omega_\mu$, let 2^α denote the set of all functions from α into $2 = \{0, 1\}$. We use the *ω_μ-box topology* on 2^{ω_μ} which we recall has as a base all sets of the form $[f] = \{g \in 2^{\omega_\mu} : f \subset g\}$, where f is a function from some ordinal $\alpha < \omega_\mu$ into $\{0, 1\}$ (i.e., $f \in 2^\alpha$) [4]. For $x \in 2^{\omega_\mu}$, let $x \restriction \alpha$ denote the restriction of the function x to $\alpha \subset \omega_\mu$.

For a point $q = (x, \omega_\mu) \in 2^{\omega_\mu} \times L(\omega_\mu)$, a local base in the product topology is given by $\{N(q, \alpha) = [x \restriction \alpha] \times (\alpha, \omega_\mu] : \alpha < \omega_\mu\}$. For $q = (x, \omega_\mu)$, let $C(q) = \{(x, \beta) : \beta < \omega_\mu\}$.

For the underlying set take $X = 2^{\omega_\mu} \times L(\omega_\mu)$, and for the topology on X take the one in which every point of the form (x, γ) for $\gamma < \omega_\mu$ is isolated and a point $q = (x, \omega_\mu)$, has as a local base all sets of the form $U(q, \alpha) = N(q, \alpha) \setminus C(q)$ for $\alpha < \omega_\mu$.

Example 4.1 *The space X is an ω_μ-Nagata space and does not have an ortho-base.*

Proof. It is routine to check that the described neighborhoods generate a T_2-topology on $2^{\omega_\mu} \times L(\omega_\mu)$ that is finer that the product topology on $2^{\omega_\mu} \times L(\omega_\mu)$. Moreover, for each point $p = (x, \omega_\mu)$ the family $\{U(p, \alpha) : \alpha < \omega_\mu\}$ forms a decreasing local base of clopen sets indexed by ω_μ. Hence X is ω_μ-additive.

Claim: X is ω_μ-Nagata. Since every point has a decreasing local base indexed by ω_μ, it suffices to show that X is stratifiable over ω_μ. Given α, U define

$$S(\alpha, U) = U \cap (2^{\omega_\mu} \times [0, \alpha]) \cup \bigcup\{U(x, \omega_\mu, \alpha) : U(x, \omega_\mu, \alpha) \subset U\}.$$

Since the other properties are clear, we show that $S(\alpha, U)$ is clopen, hence S satisfies LS_1. To prove this it suffices to consider $(y, \omega_\mu) \in cl_X(S(\alpha, U))$. Then $U(y, \omega_\mu, \alpha) \cap S(\alpha, U) \neq \emptyset$ so there exists $U(x, \omega_\mu, \alpha) \subset U$ such that $U(y, \omega_\mu, \alpha) \cap U(x, \omega_\mu, \alpha) \neq \emptyset$. This implies that $y \restriction \alpha = x \restriction \alpha$, hence $(y, \omega_\mu) \in U(x, \omega_\mu, \alpha) \subset S(\alpha, U)$.

Claim: X does not have an ortho-base. Let $\mathcal{B}$ be any base for X. We will find a non-isolated point p and $\mathcal{B}' \subset \mathcal{B}$ such that $\{p\} = \cap\mathcal{B}'$, but $\mathcal{B}'$ is not a local base for p.

We use transfinite induction on ω_μ. Assume we have constructed points $p_\tau = (x_\tau, \omega_\mu)$, sets $B_\tau \in \mathcal{B}$, and ordinals $\alpha_\tau < \omega_\mu$ for all $\tau < \gamma$ satisfying the following properties for all $\sigma < \gamma$

(1) $\sup\{\alpha_\tau : \tau < \sigma\} < \alpha_\sigma$,
(2) $U(p_\sigma, \alpha_\sigma) \subset B_\sigma \subset \bigcap\{U(p_\mu, \alpha_\mu) : \mu < \sigma\}$,
(3) if $\tau < \sigma$, then there exists $\alpha_\tau < \beta < \alpha_\sigma$ with $x_\tau(\beta) \neq x_\sigma(\beta)$.

We construct step γ as follows: Let $\rho = \sup\{\alpha_\tau : \tau < \gamma\}$. Then $f = \cup\{x_\tau \restriction \alpha_\tau : \tau < \gamma\}$ is a function (in 2^ρ). To see this, note that if $\mu < \sigma < \gamma$, then by (1) $\alpha_\mu < \alpha_\sigma$ hence by (2) $x_\sigma \in [x_\mu \restriction \alpha_\mu]$; so $x_\mu \restriction \alpha_\mu = x_\sigma \restriction \alpha_\mu$. We define x_γ by cases: $x_\gamma(\eta) = f(\eta)$ if $\eta < \rho$; $x_\gamma(\eta) = 1$ if $\rho \leq \eta < \rho + \rho$ and $x_\gamma(\eta) = 0$ if $\eta \geq \rho + \rho$. The first case of the definition shows that $(x_\gamma, \omega_\mu) \in U(x_t, \omega_\mu, \alpha_\tau)$ for all $\tau < \gamma$, hence we may pick B_γ and $\alpha_\gamma > \rho + \rho$ such that $p_\gamma \in U(p_\gamma, \alpha_\gamma) \subset B_\gamma \subset \cap\{U(p_\tau, \alpha_\tau) : \tau < \gamma\}$. Note that by this construction $x_\tau(\eta) = 0$ for $\eta \geq \alpha_\tau$, so (3) holds.

Now we define $x = \cup\{x_\tau \restriction \alpha_\tau : \tau < \omega_\mu\}$. Then $x \in 2^{\omega_\mu}$. Put $p = (x, \omega_\mu)$, and $\mathcal{B}' = \{B_\alpha : \alpha < \omega_\mu\}$. We check that $p \in \cap\mathcal{B}'$: for any $\sigma < \omega_\mu$, by (2) $p_\sigma \in U(p_\sigma, \alpha_\sigma) \subset B_\sigma$, and since $x_\sigma \restriction \alpha_\sigma = x \restriction \alpha_\sigma$, we have $p \in U(p_\sigma, \alpha_\sigma) \subset B_\sigma$. Now if $q \in \cap\mathcal{B}'$ then q must be a non-isolated point since $\sup\{\alpha_\sigma : \sigma < \omega_\mu\} = \omega_\mu$. But then $q = (y, \omega_\mu)$ and $q \in U(p_\sigma, \alpha_\sigma)$ imply $y \restriction \alpha_\sigma = x \restriction \alpha_\sigma$ for all $\sigma < \omega_\mu$; so $y = x$, and $q = p$. To complete the proof we show that $\{B_\alpha : \alpha < \omega_\mu\}$ is not a local base at p. In fact, $B_\sigma \not\subset U(p, 0)$ for any $\sigma < \omega_\mu$. To see this, let $\sigma < \omega_\mu$ be given. By (3) there exists $\beta > \alpha_\sigma$ such that $x(\beta) \neq x_\sigma(\beta)$, hence $x \neq x_\sigma$; so $p \neq p_\sigma$. Hence $(x, \beta) \in U(p_\sigma, \sigma_\alpha) \subset B_\sigma$, but $(x, \beta) \in C(p)$; so $B_\sigma \not\subset U(p, 0)$.

Corollary 4.2 *There exist ω_μ-Nagata spaces that are not non-Archimedean.*

Corollary 4.3 *There exists a Nagata space that does not have an ortho-base.*

5 Spaces stratifiable over a set of regular cardinals

Since a space can be stratifiable over more than one cardinal, it is natural to ask about the set of all regular cardinals ω_μ over which a space is stratifiable. We consider the following version of that question. The restriction to regular cardinals stems from the property that a space is stratifiable over a singular cardinal λ if and only if it is stratifiable over $\mathrm{cf}(\lambda)$.

Question 5.1 *If S is a set of regular cardinals, does there exists a space $X(S)$ such that for every regular cardinal κ, $X(S)$ is stratifiable over κ if and only if $\kappa \in S$?*

We were not able to answer this question completely, but we give an affirmative answer assuming there do not exist any inaccessible cardinals (see Corollary 5.9). The spaces $X(S)$ that we construct are rather simple in that they have exactly one non-isolated point. Not all such spaces are linearly stratifiable.

Example 5.2 *There exists a space with exactly one non-isolated point which is not linearly stratifiable.*

Proof. Start with the topological disjoint union $L(\omega) \oplus L(\omega_1)$ (defined in §4) and let X be the quotient space obtained by collapsing the closed set $\{\omega, \omega_1\}$ to a point. Thus X has a convergent sequence, hence cannot be stratifiable over any uncountable regular cardinal [22, 2.10], and an open set which is not a countable union of closed sets, hence X cannot be stratifiable over ω_0 ([2, Theorem 2.2] or [22, 4.1 A]). Thus X is not linearly stratifiable. Likewise, $L(\omega) \oplus L(\omega_1)$ is not linearly stratifiable (this example was used by Nyikos and Reichel as an example of a non-Archimedean space that is not ω_μ-metrizable [19, Example 20]).

Notation 5.3 *Let S be a set of regular cardinal numbers. Let $\prod S$ denote the usual Cartesian product (i.e., $\prod S$ is the set of all functions f with domain S and satisfying $f(s) < s$ for all $s \in S$), and let $\sigma\prod S$ denote $\{f \in \prod S : \{s \in S : f(s) \neq 0\}$ is finite$\}$.*

Define the space $Z_0(S) = \prod S \cup \{p\}$ where p is an arbitrary point not in $\prod S$. We define the topology on $Z_0(S)$ as follows: all points $f \in \prod S$ are isolated, and basic neighborhoods of p are of the form $U_g = \{f \in Z_0(S) : (\forall s \in S)(g(s) \leq f(s))\} \cup \{p\}$, for $g \in \prod S$

Define the space $Z_1(S) = \sigma\prod S \cup \{p\}$ where p is an arbitrary point not in $\sigma\prod S$. We define the topology on $Z_1(S)$ as follows: all points $f \in \sigma\prod S$ are isolated, and basic neighborhoods of p are of the form $U_g = \{f \in Z_1(S) : (\forall s \in S)(g(s) \leq f(s))\} \cup \{p\}$, for $g \in \sigma\prod S$.

The next result generalizes [22, Example 7.3].

Lemma 5.4 *Both $Z_0(S)$ and $Z_1(S)$ are stratifiable over every $s \in S$. For every $\kappa < \min S$, $Z_0(S)$ is not stratifiable over κ. For $\kappa > \sup S$, $Z_1(S)$ is not stratifiable over κ.*

Proof. Fix $s \in S$. For every $\alpha < s$ and $i \in 2$ define the sets $X^i_\alpha = \{f \in Z_i(S) : f(s) = \alpha\}$. For every $\alpha < s$, the sets $\cup\{X^i_\beta : \beta \leq \alpha\}$ are closed in $Z_i(S)$ ($i \in 2$, respectively). In both cases a stratification can be defined by $S(\alpha, U) = U$ if $p \in U$, and $S(\alpha, U) = U \cap (\cup\{X^i_\beta : \beta \leq \alpha\})$ if $p \notin U$.

Now let $\kappa < \min S$. We show that $Z_0(S)$ is not stratifiable over κ. By way of contradiction, assume that $Z_0(S)$ is stratifiable over κ. Then by LS_1, LS_2 there exists a decreasing family of open sets $\{G_\alpha : \alpha < \kappa\}$ such that $\{p\} = \cap\{G_\alpha : \alpha < \kappa\}$. There exists g_α such that $U(g_\alpha) \subset G_\alpha$ for all $\alpha < \kappa$. For $s \in S$, define $h(s) = \sup\{g_\alpha(s) : \alpha < \kappa\} + 1$. Then $h(s) < s$ since s is regular and $\kappa < s$, $h \in Z_0(S) \setminus \{p\}$ and

$$h \in \cap\{U(g_\alpha) : \alpha < s\} \subset \cap\{G_\alpha : \alpha < \kappa\} = \{p\},$$

a contradiction.

Now let $\kappa > \sup S$. It is easy to see that $Z_1(S)$ is not stratifiable over κ because $|Z_1(S)| = \sup S < \kappa$. By [22, Prop. 2.10] in any space stratifiable over a regular cardinal κ, every subset of cardinality less than κ is closed discrete. But this would contradict that $Z_1(S)$ has a non-isolated point. This completes the proof.

By $\overline{S}$ we mean the closure of S in the usual order topology on $\lambda = \sup S + 1$.

Lemma 5.5 *Let S be a set of regular cardinals, and κ a regular cardinal such that $\kappa \notin \overline{S}$ and $\min S < \kappa \leq \sup S$. Then there exists a space $Z(S, \kappa)$ that has exactly one non-isolated point, and is stratifiable over every $s \in S$, and not stratifiable over κ.*

Proof. Put

$$Z(S,\kappa) = \{f \in \prod S : \{s \in S \cap \kappa : f(s) \neq 0\} \text{ is finite }\} \cup \{p\},$$

where p is an arbitrary point not in $\prod S$. To define the topology on $Z(S,\kappa)$, take all $f \in Z(S,\kappa) \setminus \{p\}$ to be isolated, and define basic neighborhoods of p for $g \in Z(S,\kappa) \setminus \{p\}$ by $U_g = \{f \in Z(S,\kappa) \setminus \{p\} : (\forall s \in S)(g(s) \leq f(s))\}$. The space $Z(S,\kappa)$ can be seen to be stratifiable over every $s \in S$ in the same way as for $Z_0(S)$ and $Z_1(S)$. We show that $Z(S,\kappa)$ is not stratifiable over κ. By way of contradiction, assume that $Z(S,\kappa)$ is stratifiable over κ. Then there exists a decreasing family of open sets $\{G_\alpha : \alpha < \kappa\}$ such that $\{p\} = \cap\{G_\alpha : \alpha < \kappa\}$. There exists g_α such that $U(g_\alpha) \subset G_\alpha$ for all $\alpha < \kappa$. Since κ is regular, and $|\sigma \prod S \cap \kappa| = \sup S \cap \kappa < \kappa$ (by hypothesis), there exist $x \in \{f \in \prod S \cap \kappa : \{s \in S \cap \kappa : f(s) \neq 0\} \text{ is finite }\}$ and $A \in [\kappa]^\kappa$ such that for all $\alpha \in A$, $g_\alpha \restriction (S \cap \kappa) = x$. For $s \in S \setminus \kappa$, define $h(s) = \sup\{g_\alpha(s) : \alpha < \kappa\}$. Then $h(s) < s$ since s is regular. Define

$$y(s) = \begin{cases} x(s) & \text{if } s \in S \cap \kappa \\ h(s) & \text{if } s \in S \setminus \kappa. \end{cases}$$

Then $y \in Z(S,\kappa) \setminus \{p\}$ and

$$y \in \cap\{U(g_\alpha) : \alpha \in A\} \subset \cap\{G_\alpha : \alpha \in A\} = \cap\{G_\alpha : \alpha < \kappa\} = \{p\}$$

where the first equality follows because the G_α are decreasing. This contradiction completes the proof

Now we prove the main result of this section.

Theorem 5.6 *For every set S of regular cardinals, there exists a space $X(S)$, having exactly one non-isolated point, such that $X(S)$ is stratifiable over every $\kappa \in S$. Moreover, $X(S)$ is not stratifiable over any regular cardinal $\kappa \notin \overline{S}$.*

Proof. Let $Y(S)$ denote the following disjoint union

$$Y(S) = Z_0(S) \oplus Z_1(S) \oplus (\oplus\{Z(S,\kappa) : \min S \leq \kappa \leq \sup S \text{ and } \kappa \notin \overline{S})\}.$$

It is easy to see that for every $s \in S$, $Y(S)$ is stratifiable over s. Now let κ be a regular cardinal, and $\kappa \notin \overline{S}$. For $\kappa < \min S$ or $\kappa > \sup S$, $Y(S)$ is not stratifiable over κ by Lemma 5.4 and heredity. Now let $\min S < \kappa \leq \sup S$ and $\kappa \notin \overline{S}$. Thus by Lemma 5.5, $Z(S,\kappa)$, and hence $Y(S)$, is not stratifiable over κ. Let $X(S)$ be the quotient space obtained by collapsing the set of non-isolated points in Y to a single point. Since the quotient map is a closed map, $X(S)$ is stratifiable over s for all $s \in S$ [22, 4.1 D], and by heredity $X(S)$ is not stratifiable over any cardinal $\kappa \notin \overline{S}$. This completes the proof.

Corollary 5.7 *If S is a set of regular cardinals such that $S = \overline{S}$ and $X(S)$ is the space defined in Theorem 5.6 then for every regular cardinal κ, $X(S)$ is stratifiable over κ if and only if $\kappa \in S$.*

In case S is a finite set of regular cardinals, $S = \overline{S}$; so the preceding corollary can be applied to S. A simpler example, however, can be given for finite S: The techniques already discussed also show that for S finite, $Z_0(S)$ is stratifiable over κ iff $\kappa \in S$.

Corollary 5.8 *For every ω_μ there exists an ω_μ-stratifiable space X that is not κ-Nagata for any cardinal κ. Moreover there exists such a space that is stratifiable.*

Proof. Let $S = \{\omega_0, \omega_\mu\}$. By Corollary 5.7, $X(S)$ is stratifiable over both ω_0 and ω_μ (but not stratifiable over any other cardinals). Since $X(S)$ is stratifiable over two regular cardinals, it is not κ-Nagata for any cardinal κ. We could also use $Z_0(S)$ since S is finite.

Corollary 5.9 *Assume there exist no inaccessible cardinals. Then for every set S of regular cardinals, there exists a space $X(S)$ having exactly one non-isolated point such that for every regular cardinal κ, $X(S)$ is stratifiable over κ if and only if $\kappa \in S$.*

We mention a special case of Question 5.1

Question 5.10 *Let κ be an inaccessible cardinal, and S a set of regular cardinals cofinal in κ. Does there exist a space stratifiable over S but not over κ? Does there exist one having exactly one non-isolated point?*

Sikorski proved that if ω_μ is an uncountable regular cardinal then every regular, ω_μ-additive space of weight ω_μ is ω_μ-metrizable [21]. Thus every ω_μ-Nagata space with weight ω_μ is ω_μ-metrizable (for the countable case, the result follows from the Urysohn metrization theorem). This result does not extend to ω_μ-stratifiable spaces.

Example 5.11 *An ω_μ-stratifiable space ($\mu > 0$) of weight ω_μ that is not κ-Nagata for any κ.*

Proof. Let $S = \{\omega_0, \omega_\mu\}$. The required example is $X = Z_0(S) \times L(\omega_\mu)$ where $Z_0(S)$, and $L(\omega_\mu)$ are defined in §4. Since both spaces are stratifiable over ω_μ and have weight ω_μ, the product X has these same properties. Further, X is not stratifiable over any cardinal different from ω_μ since $L(\omega_\mu)$ is not; so ω_μ is the only cardinal over which X is stratifiable; hence X is ω_μ-stratifiable. The space X is not κ-Nagata for any κ since it contains a subspace homeomorphic to $Z_0(\{\omega_0, \omega_\mu\})$, which is not κ-Nagata for any κ.

The preceding example show a difference between the countable and uncountable cases since every stratifiable space of weight ω_0 is Nagata (in fact metrizable).

The spaces constructed in this section can also be shown to be M_1 over the respective cardinals (see [10]).

6 Box products of ω_μ-stratifiable spaces

Let $\Box X^{\omega_\mu}$ denote the product of ω_μ many copies of X with the box topology. We prove the following

Theorem 6.1 *If $S = \{\omega_\mu, \omega_{\mu+1}\}$ then $\Box X(S)^{\omega_\mu}$ is stratifiable over $\omega_{\mu+1}$ and over no other regular cardinal.*

Corollary 6.2 *If $S = \{\omega_0, \omega_1\}$ then $\Box X^{\omega_0}$ is stratifiable over ω_1 and over no other regular cardinal. In particular, $\Box X(S)^{\omega_0}$ is not stratifiable.*

This corollary may be compared with the result of E. K. van Douwen that a box product of countably many metric spaces need not be stratifiable. In fact, van Douwen showed that $\Box \mathbf{P}^\omega$, the product of countably many copies of the irrational numbers $\mathbf{P}$ with the box topology, is not stratifiable [3] by showing that it is not normal. Our example differs from van Douwen's in two ways. He starts with the metric space $\mathbf{P}$, and gets a non-normal box product $\Box \mathbf{P}^\omega$. We start with a non-metrizable space $X(S)$, but we get a product $\Box X(S)^\omega$ that is stratifiable over ω_1, hence hereditarily paracompact and monotonically normal (Theorem 3.2 and [22]).

Proof of Theorem 6.1. By [22, Theorem 5.2], $\Box X(S)^{\omega_\mu}$ is stratifiable over $\omega_{\mu+1}$. Since $X(S)$ is ω_μ-additive, so is $\Box X(S)^{\omega_\mu}$. To complete the proof it suffices to show that $\Box X(S)^{\omega_\mu}$ is not stratifiable over ω_μ. To do this we will show that there is a closed set A that is not equal to the intersection of ω_μ many open sets (thus LS_1 fails). Take

$$A = \{f \in \Box X(S)^{\omega_\mu} : |\{\alpha < \omega_\mu : f(\alpha) \neq p\}| < \omega_\mu\},$$

where p is the non-isolated point of $X(S)$. Suppose that $\{U_\alpha : \alpha < \omega_\mu\}$ is a family of open sets each containing A. We use the notation that given $x_\beta \in X(S)$ for $\beta < \gamma < \omega_\mu$, the function $f_\gamma \in A$ is defined by $f_\gamma(\alpha) = x_\alpha$ for $\alpha < \gamma$, and $f_\gamma(\alpha) = p$ otherwise. We construct by induction on ω_μ open boxes $\Pi_\alpha B^\beta_\alpha$ and points $x_\beta \in \cap\{B^\tau_\beta : \tau \leq \beta\} \setminus \{p\}$ such that

$$f_\beta \in \Pi_\alpha B^\beta_\alpha \subset U_\beta \cap (\bigcap_{\tau\leq\beta} \Pi_\alpha B^\tau_\alpha).$$

To begin let f_0 denote the constant function in $\Box X^{\omega_0}$ with constant value p. Since $f_0 \in A$, we may select an open box $\Pi_\alpha B^0_\alpha$ such that $f_0 \in \Pi_\alpha B^0_\alpha \subset U_0$. Pick $x_0 \in B^0_0 \setminus \{p\}$. At step γ, we have $f_\gamma \in A \cap \bigcap_{\beta \leq \gamma}(\Pi_\alpha B^\beta_\alpha)$. By ω_μ-additivity, we may select an open box $\Pi_\alpha B^\gamma_\alpha$ such that

$$f_\gamma \in \Pi_\alpha B^\gamma_\alpha \subset U_\gamma \cap \bigcap_{\beta \leq \gamma} (\Pi_\alpha B^\beta_\alpha),$$

and choose $x_\gamma \in \cap\{B^\tau_\gamma : \tau \leq \gamma\} \setminus \{p\}$. It follows that $x = (x_\alpha) \in \cap\{U_\alpha : \alpha < \omega_\mu\}$, and $x \notin A$.

7 Character versus pseudocharacter

It is easy to see that if X is stratifiable over ω_μ then $\psi(X) \leq \omega_\mu \leq \chi(X)$ [22, 2.12], where $\psi(X)$ denotes the pseudocharacter of X and $\chi(X)$ denotes the character of X (see [8]). It is natural to ask: can we get both inequalities to be strict? The spaces in §2 do not completely answer this question since their pseudocharacter equals the smallest cardinal over which they are stratifiable.

Example 7.1 *The space $X = \Box X(\{\omega, \omega_\mu\})^\omega \times X(\{\omega, \omega_\mu, \omega_{\mu+1}\})$ $(\mu > 0)$ is stratifiable over ω_μ (and no other regular cardinal) and satisfies $\omega = \psi(X) < \omega_\mu < \omega_{\mu+1} \leq \chi(X)$.*

Proof. All the coordinate spaces are stratifiable over ω_μ, hence X is stratifiable over ω_μ [22, Theorem 5.2]. The space $X(\{\omega, \omega_\mu\})$ is stratifiable, hence every point (in particular the non-isolated point) is a G_δ; so in the countable box product every point is a G_δ. Also $X(\{\omega, \omega_\mu, \omega_{\mu+1}\})$ is stratifiable. Thus X is the product of two spaces in which every point is a G_δ; so $\psi(X) = \omega_0$. The character of the non-isolated point in $X(\{\omega, \omega_\mu, \omega_{\mu+1}\})$ is $\omega_{\mu+1}$, hence $\chi(X) \geq \omega_{\mu+1}$.

Question 7.2 *For any regular ω_μ, does there exist a linearly stratifiable space X such that $\omega_\mu = \psi(X) < s < \chi(X)$ for all regular cardinals s over which X is stratifiable?*

There are simple examples of countable spaces that have no point of first countability. Using the same idea, and the previous results, we give the following example.

Example 7.3 *For every regular cardinal ω_μ, there exists an ω_μ-stratifiable space X such that $|X| = \omega_\mu$ and every point of X has character greater than ω_μ.*

Proof. Let Z be the quotient space obtained from the disjoint union of ω_μ copies of $L(\omega_\mu)$ by collapsing the non-isolated points to a single point, denoted by p. Then Z is stratifiable over ω_μ and has one point, p, with character greater than ω_μ. Put $Y = \Box Z^\omega$, and define $X = \{f \in Y : |\{n \in \omega : f(n) \neq p\}| < \omega\}$. By the product theorem [22, Theorem 5.2] the space Y is stratifiable over ω_μ, hence by heredity X is stratifiable over ω_μ. The space X contains homeomorphic copies of Z (e.g., map z to $(z, p, ..., p, ...)$) , hence of $L(\omega_\mu)$; thus X is not stratifiable over any cardinal other than ω_μ. The other two properties of X follow easily.

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