

Universal ultrametric space of weight τ^ω

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Abstract: We give an affirmative solution to a problem raised by A. J. Lemin, by showing that any ultrametric space of weight at most τ^ω can be isometrically embedded into the Lemins' ultrametric space (LW_τ, Δ) .

MSC: 03E10, 54A25, 54C25, 54E35.

1 Introduction

A. J. Lemin and V. A. Lemin constructed for every cardinal $\tau \geq 2$ an ultrametric space called (LW_τ, Δ) into which every ultrametric space of weight at most τ can be isometrically embedded [2, Main Theorem] (such a space is called τ -universal). The weight of (LW_τ, Δ) is τ^ω , and this fact raises the natural question: [1, Problem 2]: is (LW_τ, Δ) a τ^ω -universal space, i.e., can every ultrametric space of weight at most τ^ω be isometrically embedded into (LW_τ, Δ) ?

In this paper we solve this problem in the affirmative by proving that (LW_τ, Δ) is indeed τ^ω -universal (Theorem 1.2).

Recall that a metric space (X, d) is called an *ultrametric space* provided the metric d satisfies the Strong Triangle Inequality: For all $x, y, z \in X$

$$d(x, z) \leq \max\{d(x, y), d(y, z)\},$$

or, equivalently, satisfies the following Isosceles Property: For all $x, y, z \in X$ $\Delta_{IP}(x, y, z)$: Among the numbers $\{d(x, y), d(y, z), d(x, z)\}$ two are equal and greater than or equal to the third.

Definition 1.1 [2] *The underlying set of (LW_τ, Δ) is the set of all functions f from the set \mathbb{Q}_+ of positive rational numbers into τ , which are eventually zero (i.e., there exists a real number r such that $f(x) = 0$ for all $x > r$). The ultrametric Δ is defined by $\Delta(f, g) = \sup\{x \in \mathbb{Q}_+ : f(x) \neq g(x)\}$.*

Theorem 1.2 *Let $\tau \geq 2$. Then LW_τ is τ^ω -universal.*

We give a complete proof of Theorem 1.2 in this paper. Our proof of Theorem 1.2 follows the two cases used by the Lemins in their proof of [2, Main theorem] and differs from the Lemins' proof mainly in Case 1 where we use a new construction of certain functions in LW_τ (see Lemma 1.3 below). Our proof also differs from the Lemins' proof in that we define the isometry directly on all of X as in [3], rather than first defining it on a dense subset as in [2], and we employ a different presentation of the use of the Isosceles Property.

Lemma 1.3 *If $b \geq 0$, $h \in LW_\tau$ and $Y \subset LW_\tau$ with $|Y| < \tau^\omega$, then there exists $f \in LW_\tau$ such that $\Delta(f, h) = b$, and for all $y \in Y$, $\Delta(h, y) \geq b$.*

Proof. If $b = 0$ then $f = h$ works; so we assume $b > 0$. Put $d_0 = 0$ and for $n \geq 1$ pick an increasing sequence of positive irrational numbers (d_n) converging up to b . Let I_n denote the interval of rational numbers $(d_n, d_n + 1) \cap \mathbb{Q}_+$. Since that are τ^ω functions with domain I_n and range τ , and $|Y| < \tau^\omega$, we may pick a function g_n with domain I_n and range τ such that $g_n \neq y|_{I_n}$ for all $y \in Y \cup \{h\}$. We define

$$f = \cup\{g_n : n \in \omega\} \cup h|_{[b, \infty)}$$

Clearly, $f \in LW_\tau$, and $\Delta(f, h) \leq b$. Moreover $\Delta(f, y) \geq b$ for all $y \in Y \cup \{h\}$ because f and y differ below b arbitrarily close to b . In particular $\Delta(f, h) = b$.

2 Proof of the Theorem

Let (X, d) be an ultrametric space with weight not more than τ^ω . It follows that the cardinality of X is no more than τ^ω ; so we may assume $|X| = \tau^\omega$ and in a one-one fashion, well-order $X = \{x_\alpha : \alpha < \tau^\omega\}$. We want to isometrically embed X into (LW_τ, Δ) .

Assume we have defined $f_\alpha \in LW_\tau$ for $\alpha < \gamma$ such that the map $x_\alpha \mapsto f_\alpha$ satisfies $d(x_\alpha, x_\beta) = \Delta(f_\alpha, f_\beta)$ for $\alpha < \beta < \gamma$. We define f_γ and show that the assignment $x_\gamma \mapsto f_\gamma$ extends the isometry. First (as in [2]) define

$$d_\gamma = \inf\{d(x_\alpha, x_\gamma) : \alpha < \gamma\}.$$

Case 1: $d(x_\gamma, x_\beta) = d_\gamma$ for some $\beta < \gamma$. Let $Y = \{f_\alpha : \alpha < \gamma\}$. Thus $|Y| < \tau^\omega$; so by Lemma 1.3 there exists $f_\gamma \in LW_\tau$ such that $\Delta(f_\gamma, f_\beta) = d_\gamma$, and $\Delta(f_\gamma, f_\alpha) \geq d_\gamma$ for all $\alpha < \gamma$. To see that the assignment $x_\gamma \mapsto f_\gamma$ extends the isometry, we fix $\alpha < \gamma$ and show that $d(x_\gamma, x_\alpha) = \Delta(f_\gamma, f_\alpha)$. We may rename the following two sets of distances as indicated

$$\{d(x_\gamma, x_\beta), d(x_\beta, x_\alpha), d(x_\gamma, x_\alpha)\} = \{d_\gamma, a, b\}$$

$$\{\Delta(f_\gamma, f_\beta), \Delta(f_\beta, f_\alpha), \Delta(f_\gamma, f_\alpha)\} = \{d_\gamma, a, b'\}$$

because the first number in each set equals d_γ and the two middle numbers are equal by the Induction Hypothesis. We want to show that $b = b'$. If $a < d_\gamma$ then by $\Delta_{IP}(x_\gamma, x_\beta, x_\alpha)$, $b = d_\gamma$ and by $\Delta_{IP}(f_\gamma, f_\beta, f_\alpha)$, $b' = d_\gamma$; so $b = b'$. Similarly, if $d_\gamma < a$ then $a = b$ and $a = b'$; so $b = b'$. Finally, if $d_\gamma = a$, it does not follow in general that $b = b'$, but in our case we also know that $d_\gamma \leq b$ by definition of d_γ , and $d_\gamma \leq b'$ by Lemma 1.3. Thus by the Isosceles Property we have $d_\gamma = a = b$ and $d_\gamma = a = b'$; so $b = b'$. Hence $d(x_\gamma, x_\alpha) = \Delta(f_\gamma, f_\alpha)$.

Case 2. $d_\gamma < d(x_\gamma, x_\alpha)$ for all $\alpha < \gamma$. Thus for every $\epsilon > 0$ there exists $\alpha < \gamma$ such that $0 < d(x_\gamma, x_\alpha) - d_\gamma < \epsilon$. Pick $\alpha_1 < \gamma$ such that $d(x_\gamma, x_{\alpha_1}) < d_\gamma + 1$. Continue by induction to pick for each $i > 1$ an ordinal $\alpha_i < \gamma$ such that

$$d(x_\gamma, x_{\alpha_i}) < \min\left\{\frac{1}{i}(d_\gamma + 1), \min\{d(x_\gamma, x_{\alpha_j}) : j < i\}\right\}$$

Put $b_i = d(x_\gamma, x_{\alpha_i})$ for all i . Then (b_i) is a strictly decreasing sequence of real numbers converging to d_γ . From $x_{\alpha_i} \mapsto f_{\alpha_i}$, define

$$f_\gamma = \cup\{f_{\alpha_i} \upharpoonright (b_i, \infty) : i \geq 1\} \cup \{(x, 0) : x \in \mathbb{Q}_+ \text{ and } x \leq d_\gamma\}.$$

To see that f_γ is a well defined function and an element of LW_τ it suffices to show that $i < k$ implies $f_{\alpha_i}(x) = f_{\alpha_k}(x)$ for all $x > b_i$, hence to show $\Delta(f_{\alpha_i}, f_{\alpha_k}) \leq b_i$. To see this note that $\Delta_{IP}(x_{\alpha_i}, x_{\alpha_k}, x_\gamma)$ implies $d(x_{\alpha_i}, x_{\alpha_k}) = b_i$, hence by the induction hypothesis we have in fact $\Delta(f_{\alpha_i}, f_{\alpha_k}) = d(x_{\alpha_i}, x_{\alpha_k}) = b_i$. Thus f_γ is well defined, and clearly by definition $\Delta(f_\gamma, f_{\alpha_i}) \leq b_i$ for all i .

To see that the assignment $x_\gamma \mapsto f_\gamma$ extends the isometry, we fix $\alpha < \gamma$ and show that $d(x_\gamma, x_\alpha) = \Delta(f_\gamma, f_\alpha)$. Choose i so large that $b_i = d(x_\gamma, x_{\alpha_i}) < d(x_\gamma, x_\alpha)$. We may rename the following two sets of distances as indicated

$$\{d(x_\gamma, x_\alpha), d(x_\alpha, x_{\alpha_i}), d(x_\gamma, x_{\alpha_i})\} = \{c, a, b_i\}$$

$$\{\Delta(f_\gamma, f_\alpha), \Delta(f_\alpha, f_{\alpha_i}), \Delta(f_\gamma, f_{\alpha_i})\} = \{f, a, g\}$$

because the last number in the first set equals b_i , and the two middle numbers are equal by the induction hypothesis. We chose i so that $b_i < c$ and hence by $\Delta_{IP}(x_\gamma, x_\alpha, x_{\alpha_i})$, $c = a$. We also know $g \leq b_i < a$, hence by $\Delta_{IP}(f_\gamma, f_\alpha, f_{\alpha_i})$, $f = a$. Thus $c = f$, i.e., $d(x_\gamma, x_\alpha) = \Delta(f_\gamma, f_\alpha)$, and this completes the proof.

Remark 2.1 *Our result that (LW_τ, Δ) is τ^ω -universal, renews interest in a problem raised by the Lemins [2, Problem 1]: does there exist a τ -universal ultrametric space of weight τ ? In [3] we gave a consistent, affirmative solution to this problem (we assumed the singular cardinal hypothesis), and in ZFC an affirmative solution for a proper class of cardinals τ of countable cofinality. The problem, however, is not completely solved in ZFC.*

References

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