Two step iteration of almost disjoint families [1]

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1. Introduction

Let E be an infinite set, and $[E]^\omega$ the set of all countably infinite subsets of E. A family $\mathcal{A} \subset [E]^\omega$ is said to be *almost disjoint* (respectively, *pairwise disjoint*) provided for $A, B \in \mathcal{A}$, if $A \neq B$ then $A \cap B$ is finite (respectively, $A \cap B$ is empty). Moreover, an infinite family $\mathcal{A}$ is said to be a *maximal almost disjoint family* provided it is an infinite almost disjoint family not properly contained in any almost disjoint family. In this paper we are concerned with the following set of topological spaces defined from (maximal) almost disjoint families of infinite subsets of the natural numbers ω.

Definition 1.1. *For an almost disjoint family $\mathcal{A}_0 \subset [\omega]^\omega$, define a topological space $\psi(\mathcal{A}_0) = \omega \cup \mathcal{A}_0$ with the topology in which for every natural number $n \in \omega$, the singleton set $\{n\}$ is a local base at n (i.e., n is an isolated point), and each $A \in \mathcal{A}_0$ has a local base consisting of sets of the form $\{A\} \cup A \setminus F$ where F is a finite subset of ω.*

The space $\psi(\mathcal{A}_0)$ is well known. If the almost disjoint family is denoted by $\mathcal{R}$ and the natural numbers by $\mathcal{N}$ then the space $\psi(\mathcal{A}_0)$ was called $\mathcal{N} \cup \mathcal{R}$ by S. Mrówka [4]. The same space was called Ψ in [2, Exercise 5I], where it is attributed to J. Isbell.

All the results in this paper can be stated in terms of the spaces $\psi(\mathcal{A}_0)$ (see Remark 2.1) but for reasons of convenience and motivation (see §2), we prefer to consider $\psi(\mathcal{A}_0)$ as a subspace of a larger space, which we now define.

Definition 1.2. *Given $\psi(\mathcal{A}_0)$, let $\mathcal{A}_1 \subset [\mathcal{A}_0]^\omega$ be a maximal almost disjoint family. Define a topological space $\psi(\mathcal{A}_0, \mathcal{A}_1) = \psi(\mathcal{A}_0) \cup \mathcal{A}_1 = \omega \cup \mathcal{A}_0 \cup \mathcal{A}_1$ with the topology in which local bases for points in $\omega \cup \mathcal{A}_0$ are taken to be the same as in $\psi(\mathcal{A}_0)$, and a local base for a point $X \in \mathcal{A}_1$ consists of all sets of the form*

$$\{X\} \cup (X \setminus G) \cup (\cup\{A \setminus F(A) : A \in X \setminus G\})$$

where G is finite, and $F(A)$ is finite for all $A \in X \setminus G$.

The space $\psi(\mathcal{A}_0, \mathcal{A}_1)$, called a *two step iteration of ψ*, was introduced in [6].

Definition 1.3. *A topological space X satisfies the* Hausdorff *(respectively,* Urysohn*)* separation property (or axiom) *provided for every pair of distinct points $x, y \in X$ there exit open sets U, V in X such that $x \in U, y \in V$ and $U \cap V = \emptyset$ (respectively $cl(U) \cap cl(V) = \emptyset$), where the* closure *of a set $S \subset X$ is defined by $cl(S) = \{x \in X :$ for every open set V containing x, $V \cap S \neq \emptyset\}$.*

In [6] we proved that every $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is a Hausdorff space, i.e, satisfies the Hausdorff property (see Corollary 4.2). The basic question we consider here is this:

Question 1.4. *Do there exist maximal almost disjoint families $\mathcal{A}_0, \mathcal{A}_1$ such that $\psi(\mathcal{A}_0, \mathcal{A}_1)$ satisfies the Urysohn property?*

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Our main results are the following (small uncountable cardinals are reviewed in §3). Theorem 1.5 and a preliminary version of Theorem 1.7 were presented in 1992 [13].

Theorem 1.5. *There exists a maximal $\mathcal{A}_0$ such that for every maximal $\mathcal{A}_1$, $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is not Urysohn.*

Theorem 1.6. *Assume $\mathfrak{d} \leq \mathfrak{a}$. For every maximal $\mathcal{A}_0$ there exist maximal $\mathcal{A}_1$ such that $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is not Urysohn.*

Theorem 1.7. *Assume $\mathfrak{h} = \mathfrak{c}$. There exist maximal $\mathcal{A}_0$ and $\mathcal{A}_1$ such that $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is Urysohn.*

Theorem 1.5 and Theorem 1.6 show that in order to obtain a $\psi(\mathcal{A}_0, \mathcal{A}_1)$ that is Urysohn, a careful construction of $\mathcal{A}_0$ and $\mathcal{A}_1$ is needed (indeed $\mathfrak{h} = \mathfrak{c}$ implies $\mathfrak{d} = \mathfrak{a}$). Theorem 1.7 provides a consistent, affirmative answer to Question 1.4, but at least one step in our proof cannot be carried out in ZFC alone (see Corollary 7.2); so Question 1.4 remains open in ZFC.

2. Motivation

Our interest in Question 1.4 comes from the Scarborough-Stone problem [8]: Is every product of sequentially compact spaces countably compact? At this time the only complete solutions to this problem are for the class of hereditarily normal (i.e., T_5-spaces) and the class of Hausdorff spaces. For T_5-spaces there is a consistent, positive solution [7] and a consistent negative solution [11]; so the problem is independent and consistent for T_5-spaces. For the class of Hausdorff spaces, there is in ZFC a negative solution, i.e., a counterexample to the problem [6].

This brings up the question of whether there is in ZFC a counterexample to the Scarborough-Stone problem in the class of Urysohn spaces, a separation axiom which is obviously stronger than Hausdorff (and weaker than regular, i.e, T_3). A more specific question: can a Urysohn counterexample be constructed using an iteration of ψ, as was done in [6] to produce a Hausdorff counterexample? To use the iteration of ψ as in [6], one must first ask whether $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is Urysohn because $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is the second step of the iteration (this is Question 1.4). While we do not completely answer these questions, the results in this paper indicate that the spaces $\psi(\mathcal{A}_0, \mathcal{A}_1)$ have interesting connections with fundamental properties of the natural numbers, and are of some interest in themselves.

Remark 2.1. *If $\mathcal{A}_0$ is maximal, and $\mathcal{A}_1 \neq \emptyset$, then $\psi(\mathcal{A}_0, \mathcal{A}_1)$ not regular. Nevertheless, the results in this paper can be interpreted as results about $\psi(\mathcal{A}_0)$ which is completely regular (in fact, a zero dimensional Hausdorff space). This holds because one may work with $\mathcal{A}_1$, a family of subsets of $\psi(\mathcal{A}_0)$, without reference to a larger space.*

3. Small uncountable cardinals

We recall some definitions (see [1] and [12]). Let ω denote the set of natual numbers, ${}^\omega\omega$ the set of all functions $f : \omega \to \omega$, and $[\omega]^\omega$ the set of all infinite subsets of ω. We consider two partial orders on ${}^\omega\omega$. Define $f \leq g$ provided $f(i) \leq g(i)$ for all $i \in \omega$. We call this the *coordinate-wise order*. Define $f \leq^* g$ provided there exists $N \in \omega$ such that $f(i) \leq g(i)$ for all $i \geq N$. We call this the *mod finite order* (or the *order of eventual domination*). A set $X \subset {}^\omega\omega$ is *dominating* in the coordinate-wise order (respectively, the mod finite order) provided for every $f \in {}^\omega\omega$ there exists $g \in X$ such that $f \leq g$ (respectively, $f \leq^* g$). We also consider the order on $[\omega]^\omega$ defined by $A \subset^* B$ provided $A \setminus B$ is finite.

$\mathfrak{a} = \min\{|A| : A \subset [\omega]^\omega$ is an infinite, maximal almost disjoint family $\}$.

$\mathfrak{b} = \min\{|B| : B \subset {}^\omega\omega$ is unbounded in the mod finite order$\}$.

$\mathfrak{c} = |[\omega]^\omega|$ (the cardinality of the continuum).

$\mathfrak{d} = \min\{|D| : D \subset {}^\omega\omega$ is dominating in the mod finite order$\}$.

Recall that $\mathfrak{d}$ is also the minimum cardinality of a set $D \subset {}^\omega\omega$ that is dominating in the coordinate-wise order [1, 3.6]

A family $D \subset [\omega]^\omega$, is *dense* provided for every infinite $A \in [\omega]^\omega$ there is $K \in D$ such that $K \subset^* A$, and D is called *open* provided for each $K \in D$ and $H \subset^* K$, $H \in D$.

$\mathfrak{h} = \min\{|\mathcal{D}| : \mathcal{D}$ is a family of dense open sets and $\cap \mathcal{D} = \emptyset\}$.

We use the well known fact that if $\mathcal{D}$ is a family of dense open sets and $|\mathcal{D}| < \mathfrak{h}$ then $\cap\mathcal{D}$ is dense. A family $\mathcal{R} \subset [\omega]^\omega$ is called a *refining family* on ω provided for every $H \in [\omega]^\omega$ there exists $R \in \mathcal{R}$ such that $R \subset H$ or $R \subset \omega \setminus H$.

$\mathfrak{r} = \min\{|\mathcal{R}| : \mathcal{R} \subset [\omega]^\omega$ is a refining family on $\omega\}$

A set H *splits* a set K provided $|K \cap H| = |K \setminus H| = \omega$. A family $\mathcal{S} \subset [\omega]^\omega$ is called *splitting* if for every $K \in [\omega]^\omega$, there exists $H \in \mathcal{S}$ such that H splits K.

$\mathfrak{s} = \min\{|S| : S \subset [\omega]^\omega$ is a splitting family on $\omega\}$.

Further recall that all six cardinals are uncountable and not larger than $\mathfrak{c}$, and $\mathfrak{h} \leq \mathfrak{b}, \mathfrak{s}$ and $\mathfrak{b} \leq \mathfrak{d}, \mathfrak{a}$. Concerning the cardinals $\mathfrak{a}$ and $\mathfrak{d}$ in the hypothesis of Theorem 1.6, we mention that they are not related to each other in general: Cohen models satisfy the inequality "$\mathfrak{a} < \mathfrak{d}$" (cf [1, Theorem 5.2]) and S. Shelah constructed a model for "$\mathfrak{d} < \mathfrak{a}$" [9].

Concerning the cardinality of $\psi(\mathcal{A}_0)$, there is a model due to S. Hechler [3], in which there exist infinite maximal almost disjoint families of infinite subsets of ω of every cardinality κ for $\omega_1 \leq \kappa \leq \mathfrak{c}$, with $\mathfrak{c}$ arbitrarily large.

4. Notation and Preliminary Results

We relate the topology on $\psi(\mathcal{A}_0, \mathcal{A}_1)$ to ${}^\omega\omega$, the set of all functions from the natural numbers into itself. We assign a listing for each $X \in [\mathcal{A}_0]^\omega$, $X = \{X_i : i \in \omega\}$, and we pick a function $f_X \in {}^\omega\omega$ with the property $f_X(i) > \max(X_i \cap (\cup_{j<i} X_i))$ for all $i \in \omega$. Note that if $f \geq f_X$, then $X_j \cap (X_i \setminus f(i)) = \emptyset$ for $j < i$, hence $\{X_i \setminus f(i) : i \in \omega\}$ is a pairwise disjoint family. Since $\mathcal{A}_0$ is an almost disjoint family, for every $X \in [\mathcal{A}_0]^\omega$ there is such a function f_X. For any $X \in [\mathcal{A}_0]^\omega$, $N \in \omega$, and $f \in {}^\omega\omega$ we define

$$X \setminus N = \{X_i : i \geq N\}, \text{ and } (X \setminus N) \uparrow f = \cup\{X_i \setminus f(i) : i \geq N\}.$$

From the previous discussion, it is clear that sets of the form

$$U(X, N, f) = \{X\} \ \cup (X \setminus N) \cup (X \setminus N) \uparrow f$$

where $N \in \omega$ and $f \geq f_X$ (f increasing), form a local base at X in $\psi(\mathcal{A}_0, \mathcal{A}_1)$.

To illustrate this notation, we prove the following results. .

Lemma 4.1. *For $X, Y \in [\mathcal{A}_0]^\omega$ with $X \cap Y$ finite, there exists N, f such that $U(Y, N, f) \cap U(X, 0, f) = \emptyset$.*

Proof. Pick N large enough that $(Y \setminus N) \cap X = \emptyset$, and define for $i \geq N$

$$f(i) > \max[(\cup\{X_j : j \leq i\}) \cap (\cup\{Y_j : N \leq j \leq i\})]$$

(define $f(i) = 0$ for $i < N$). If $n \in U(Y, N, f) \cap U(X, 0, f)$, then there exist $i \geq N, j \geq 0$ such that $n \in (Y_i \setminus f(i)) \cap (X_j \setminus f(j))$, but this is impossible by the definition of f.

Corollary 4.2 ([6]). *Every $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is a Hausdorff space.*

Proof. The only separation for pairs of points that is not obvious is for two points $X, Y \in \mathcal{A}_1$, and they can be separated by disjoint open sets by Lemma 4.1.

Next, let us consider the character of points in $\psi(\mathcal{A}_0, \mathcal{A}_1)$. Recall the *character* of a point x in a space X is the smallest infinite cardinal number which is the cardinality of a local base for the point. In $\psi(\mathcal{A}_0, \mathcal{A}_1)$, points in ω are isolated (i.e., have character $\aleph_0$), points in $\mathcal{A}_0$ have a countable local base (i.e., have character $\aleph_0$), and the character of points in $\mathcal{A}_1$ is given in the next result.

Theorem 4.3. *If $\mathcal{A}_0$ is an infinite almost disjoint family, then each $X \in \mathcal{A}_1$, has character $\mathfrak{d}$ in $\psi(\mathcal{A}_0, \mathcal{A}_1)$.*

Proof. Let $D \subset {}^\omega\omega$ be dominating in the coordinate-wise order with $|D| = \mathfrak{d}$. It is clear that $\{U(X, N, f) : N \in \omega, f \in {}^\omega\omega\}$ is a local base for X of cardinality $\mathfrak{d}$. We show that if $\mathcal{B}$ is any family of neighborhoods of X and $|\mathcal{B}| < \mathfrak{d}$ then $\mathcal{B}$ is not a local base at X. Let $\mathcal{B} = \{B_\alpha : \alpha < \lambda\}$ where $\lambda < \mathfrak{d}$. For each $\alpha < \lambda$ there exists $N_\alpha \in \omega$ and $f_\alpha \geq f_X$ such that $U(X, N_\alpha, f_\alpha) \subset B_\alpha$. Define $g_\alpha(i) = min(X_i \setminus f_\alpha(i))$ for $i \in \omega$. Since $\{g_\alpha : \alpha < \lambda\}$ is not dominating (in the mod finite order) there exists an increasing $f \in {}^\omega\omega$ such that $f \not\leq^* g_\alpha$ for all $\alpha < \lambda$. Claim: No B_α in $\mathcal{B}$ is a subset of $U(X, 0, f)$, hence $\mathcal{B}$ is not a local base at X. To see that the Claim holds, let $\alpha < \lambda$. Pick $i \geq N_\alpha$ such that $g_\alpha(i) < f(i)$. Then $g_\alpha(i) \notin X_i \setminus f(i)$. Since $g_\alpha(i) \in X_i \setminus f_\alpha(i) \subset U(X, N_\alpha, f_\alpha)$, we have $g_\alpha(i) < f(i)$, and since $f_X \leq f_\alpha$, we have $g_\alpha(i) \notin X_j$ for $j < i$. Since f is increasing, $j \geq i$ implies $f(j) \geq f(i) > g_\alpha(i)$, hence $g_\alpha(i) \notin X_j \setminus f(j)$ for $j \geq i$. Thus $g_\alpha(i) \in U(X, N_\alpha, f_\alpha) \setminus U(X, 0, f)$; so $B_\alpha \not\subset U(X, 0, f)$.

Definition 4.4. *If $H \in [\omega]^\omega$ and $h : H \to \omega$, we define an extension $\bar{h}$ of h by $\bar{h}(n) = h(m)$ where m is the first integer in H such that $n \leq m$. We call $\bar{h}$ the* van Douwen extension *of h.*

5. Proof of Theorem 1.5

Start with an almost disjoint family $\mathcal{A} = \{E_\alpha : \alpha < \omega_1\} \subset [\omega]^\omega$ with $|\mathcal{A}| = \omega_1$. Partition each $E \in \mathcal{A}$ into infinitely many infinite sets $G_\alpha = \{E_{\alpha,i} : i \in \omega\}$. Define $\phi_\alpha : E_\alpha \to \omega \times \omega$ such that $\phi_\alpha \restriction E_{\alpha,i}$ is an order preserving bijection from $E_{\alpha,i}$ onto $\{i\} \times \omega$. Thus ϕ_α is a bijection. Let $\mathcal{F} = \{f_\tau : \tau < \mathfrak{b}\} \subset {}^\omega\omega$ be an unbounded chain in the mod finite order consisting of strictly increasing functions. Thus $\mathcal{F} \subset \omega \times \omega$ is an almost disjoint family; hence for each $\alpha < \omega_1$, $\{\phi_\alpha^{-1}(f_\tau) : \tau < \mathfrak{b}\}$ is an almost disjoint family. Let $\{B_\alpha : \alpha < \omega_1\}$ be a partition of $\mathfrak{b}$ where $|B_\alpha| = \mathfrak{b}$ for every $\alpha < \omega_1$. Then each set $\{f_\tau : \tau \in B_\alpha\}$ is unbounded in the mod finite order. Define

$$\mathcal{H} = \bigcup_{0<\alpha<\omega_1} \{\phi_0^{-1}(f_\tau) \cup \phi_\alpha^{-1}(f_\tau) : \tau \in B_\alpha\}.$$

To see that $\mathcal{H}$ is an almost disjoint family, consider the intersection of two of its elements

$$[\phi_0^{-1}(f_\tau) \cup \phi_\alpha^{-1}(f_\tau)] \cap [\phi_0^{-1}(f_\mu) \cup \phi_\beta^{-1}(f_\mu)].$$

Note that $\tau \neq \mu$, otherwise if $\tau = \mu$, then $\tau \in B_\alpha \cap B_\beta$, hence $\alpha = \beta$. Since both $\mathcal{A}, \mathcal{F}$ are almost disjoint, it follows that $\mathcal{H}$ is almost disjoint. Hence

$$\mathcal{A}' = \mathcal{H} \cup \{E_{\alpha,i} : \alpha < \omega_1, i < \omega\}$$

is an almost disjoint family. Let $\mathcal{A}_0$ be any (maximal or not) almost disjoint family containing $\mathcal{A}'$ and let $\mathcal{A}_1 \subset [\mathcal{A}_0]^\omega$ be a maximal almost disjoint family. To complete the proof, we show that $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is not Urysohn. Since G_0 is an infinite subset of $\mathcal{A}_0$, there exists $X \in \mathcal{A}_1$ such that $X \cap G_0$ is infinite. Note that $\{G_\alpha : \alpha < \omega_1\}$ is a pairwise disjoint family (of countable subsets of $\mathcal{A}'$) because if $S \in G_\alpha \cap G_\beta$, then $E_\alpha \cap E_\beta$ is infinite, which implies $\alpha = \beta$. Since X is countable, there exists $\alpha < \omega_1$ such that $G_\alpha \cap X = \emptyset$. Pick $Y \in \mathcal{A}_1$ such that $Y \cap G_\alpha$ is infinite. We show that $\psi(\mathcal{A}_0, \{X, Y\})$ is not Urysohn. Let $U(X, N, f), U(Y, N, f)$ be basic neighborhoods of X and Y. Let $U = U(X, N, f) \cap \omega$ and $V = U(Y, N, f) \cap \omega$. Let $J = \{j \in \omega : E_{0,j} \in X \setminus N\}$ and $K = \{k \in \omega : E_{\alpha,k} \in Y \setminus N\}$. By our choice of X, Y both J, K are infinite. Now let $\pi_2 : \omega \times \omega \to \omega$ be the usual projection map onto the second coordinate. Since $E_{0,j} \subset^* U$ and $E_{\alpha,k} \subset^* V$, we may define functions $g : J \to \omega, h : K \to \omega$ by $g(j) = \max \pi_2(\phi_0(E_{0,j} \setminus U))$ for all $j \in J$, and $h(k) = \max \pi_2(\phi_\alpha(E_{\alpha,k} \setminus V))$ for all $k \in K$. Let $\bar{g}, \bar{h}$ be the van Douwen extension of g, h respectively. Define $l(i) = \max\{\bar{g}(i), \bar{h}(i)\}$ for $i \in \omega$. There exists $\tau \in B_\alpha$ such that $f_\tau \not\leq^* l$. Put $A = \phi_0^{-1}(f_\tau) \cup \phi_\alpha^{-1}(f_\tau)$. Then $A \in \mathcal{A}' \subset \mathcal{A}_0$. We show that $A \in cl(U(X, N, f)) \cap cl(U(Y, N, f))$. It suffices to show that $|\phi_0^{-1}(f_\tau) \cap U| = |\phi_\alpha^{-1}(f_\tau) \cap V| = \omega$. To this end, let $s \in \omega$. Pick $n > \max\{s, \min J, \min K\}$ such that $l(n) < f_\tau(n)$. There exist j_1, j_2, consecutive members of J, such that $j_1 < n \leq j_2$, and k_1, k_2, consecutive members of K, such that $k_1 < n \leq k_2$. Thus

$$\max \pi_2(\phi_0(E_{0,j_2} \setminus U)) = g(j_2) = \bar{g}(n) \leq l(n) < f_\tau(n) \leq f_\tau(j_2)$$

where the last inequality holds because f_τ is an increasing function. Similarly

$$\max \pi_2(\phi_0(E_{\alpha,k_2} \setminus V)) = h(k_2) = \bar{h}(n) \leq l(n) < f_\tau(n) \leq f_\tau(k_2).$$

Thus we have $\phi_0^{-1}((j_2, f_\tau(j_2))) \notin (E_{0,j_2} \setminus U)$. Yet $(j_2, f_\tau(j_2)) \in \phi_0(E_{0,j_2})$, hence $\phi_0^{-1}((j_2, f_\tau(j_2))) \in E_{0,j_2} \cap U$. It follows that $x = \phi_0^{-1}((j_2, f_\tau(j_2))) \in U \cap \phi_0^{-1}(f_\tau)$ and since $\phi_0 \restriction E_{0,j_2}$ is order preserving, $x \geq j_2 \geq n > s$. In a similar way, $y = \phi_\alpha^{-1}(k_2, f_\tau(k_2)) \in V \cap \phi_\alpha^{-1}(f_\tau)$ and $y > s$. This completes the proof.

6. Proof of Theorem 1.7

Under several set theoretic assumptions about small uncountable cardinals, we will construct $\mathcal{A}_0, \mathcal{A}_1$, both maximal, and show that $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is Urysohn in three main steps. First we construct $\mathcal{A}_0$ such that for any $X, Y \in [\mathcal{A}_0]^\omega$ with $X \cap Y$ finite, there exist open sets U of X and V of Y such that $cl(U) \cap cl(V) \cap \mathcal{A}_0 = \emptyset$. Next, we will construct $\mathcal{A}_1$ so that $|cl(U) \cap cl(V) \cap \mathcal{A}_1| < \mathfrak{c} = \mathfrak{b}$. Then we call on

Lemma 6.4 that if $X \in \mathcal{A}_1$ and $\mathcal{B} \subset \mathcal{A}_1$ and $|\mathcal{B}| < \mathfrak{b}$ then there exists an open set W of X, such that $cl(W) \cap \mathcal{B} = \emptyset$. All the assumptions we need follow from $\mathfrak{h} = \mathfrak{c}$.

Lemma 6.1. *Assume $\mathfrak{s} = \mathfrak{c}$. There exists a maximal almost disjoint family $\mathcal{A}_0 \subset [\omega]^\omega$ such that for every $H \in [\omega]^\omega$*

$$|\{A \in \mathcal{A}_0 : |A \cap H| = |A \setminus H| = \omega\}| < \mathfrak{c}.$$

Proof. If there exists a maximal almost disjoint family$\mathcal{A}_0 \subset [\omega]^\omega$ with $|\mathcal{A}_0| < \mathfrak{c}$, then we are clearly done. Therefore we assume that $\mathfrak{a} = \mathfrak{c}$. List $[\omega]^\omega = \{H_\alpha : \alpha < \mathfrak{c}\}$, and assume that we have constructed by induction sets $A_\alpha \in [\omega]^\omega$ for $\alpha < \gamma$ (where $\gamma < \mathfrak{c}$) such that

(1) $\beta < \alpha$ implies $A_\alpha \cap A_\beta$ is finite,

(2) there exists $\beta \leq \alpha$ such that $A_\beta \cap H_\alpha$ is infinite,

(3) $\beta < \alpha$ implies $|A_\alpha \cap H_\beta| < \omega$ or $|A_\alpha \setminus H_\beta| < \omega$.

We construct A_γ as follows. First we get a set $H \in [\omega]^\omega$ such that

$$\{A_\alpha : \alpha < \gamma\} \cup \{H\}$$

is an almost disjoint family such that one of its members has infinite intersection with H_γ. To find H, take $H = H_\gamma$ if $A_\alpha \cap H_\gamma$ is finite for all $\alpha < \gamma$, otherwise, since $\mathfrak{a} = \mathfrak{c}$, $\{A_\alpha : \alpha < \gamma\}$ is not a maximal almost disjoint family; so there exists $H \in [\omega]^\omega$ such that $\{A_\alpha : \alpha < \gamma\} \cup \{H\}$ is an almost disjoint family, so take this H. By hypothesis $\mathfrak{s} = \mathfrak{c}$, so $\{H_\alpha \cap H : \alpha < \gamma\}$ is not a splitting family (on H). Thus there exists $B \in [H]^\omega$ such that no $H_\alpha \cap H$ splits B. Put $A_\gamma = B$. This completes the induction. Put $\mathcal{A}_0 = \{A_\alpha : \alpha < \omega\}$. $\mathcal{A}_0$ is a maximal almost disjoint family by (1) and (2). To see that $\mathcal{A}_0$ satisfies the conclusion of the lemma, let $H \in [\omega]^\omega$. There exists $\beta < \mathfrak{c}$ such that $H = H_\beta$. Thus by (3) for every $\beta < \alpha < \mathfrak{c}$, $|A_\alpha \cap H_\beta| < \omega$ or $|A_\alpha \setminus H_\beta| < \omega$.

Lemma 6.2. *Assume $\mathfrak{b} = \mathfrak{s} = \mathfrak{c}$. There exists maximal $\mathcal{A}_0$ such that for any $X, Y \in [\mathcal{A}_0]^\omega$, if $X \cap Y$ is finite, then there exist disjoint neighborhoods U of X and V of Y such that $cl(U) \cap cl(V) \cap \mathcal{A}_0 = \emptyset$.*

Proof. Since $\mathfrak{s} = \mathfrak{c}$, let $\mathcal{A}_0$ be a family guaranteed by Lemma 6.1. We show that $\mathcal{A}_0$ works. Let $X, Y \in [\mathcal{A}_0]^\omega$ such that $X \cap Y$ is finite. By the Hausdorff property we may find $N \in \omega$ and a function $f \in {}^\omega\omega$ such that $U(X, N, f) \cap U(Y, N, f) = \emptyset$.

Let

$$\mathcal{A}_0(X) = \{A \in \mathcal{A}_0 : |A \cap (X \setminus N) \uparrow f| = |A \setminus [(X \setminus N) \uparrow f]| = \omega\},$$

$$\mathcal{A}_0(Y) = \{A \in \mathcal{A}_0 : |A \cap (Y \setminus N) \uparrow f| = |A \setminus [(Y \setminus N) \uparrow f]| = \omega\}.$$

For every $A \in \mathcal{A}_0(X) \cup \mathcal{A}_0(Y)$, define $f_A \in [\omega]^\omega$ such that

$$A \cap [(X \setminus N) \uparrow f_A \cup (Y \setminus N) \uparrow f_A] = \emptyset.$$

Since $|\mathcal{A}_0(X) \cup \mathcal{A}_0(Y)| < \mathfrak{c} = \mathfrak{b}$ there exists $g \in [\omega]^\omega$ such that $f_A \leq^* g$ for all $A \in \mathcal{A}_0(X) \cup \mathcal{A}_0(Y)$. We may also assume that $f \leq g$. We show that

$$cl(U(X, N, g)) \cap cl(U(Y, N, g)) \cap \mathcal{A}_0 = \emptyset.$$

Let $A \in \mathcal{A}_0$. If $A \in \mathcal{A}_0(X)$, then $f_A \leq^* g$, hence $A \cap (X \setminus N) \uparrow g$ is finite; so there is a neighborhood of A missing $U(X, N, g)$. Likewise, if $A \in \mathcal{A}_0(Y)$, there is a neighborhood of A missing $U(Y, N, g)$. Otherwise, $A \notin (\mathcal{A}_0(x) \cup \mathcal{A}_0(Y))$, hence

$$|A \cap (X \setminus N) \uparrow f| < \omega \text{ or } |A \setminus [(X \setminus N) \uparrow f]| < \omega, \text{ and}$$

$$|A \cap (Y \setminus N) \uparrow f| < \omega \text{ or } |A \setminus [(Y \setminus N) \uparrow f]| < \omega.$$

The only one of the four cases that needs attention is the case

$$|A \setminus [(X \setminus N) \uparrow f]| < \omega \text{ and } |A \setminus [(Y \setminus N) \uparrow f]| < \omega.$$

This implies, however, that $A \subset^* (X \setminus N) \uparrow f$ and $A \subset^* (Y \setminus N) \uparrow f$ which is impossible since $U(X, N, f) \cap U(Y, N, f) = \emptyset$.

Lemma 6.3. *Let $K, X \in [\mathcal{A}_0]^\omega$ such that $X \subset^* K$. Then for every neighborhood U of K there exists a neighborhood V of X such that $V \setminus \{X\} \subset U$.*

Proof. Start with a basic neighborhood

$$\{K\} \cup K \setminus G \cup (\cup\{A \setminus F(A) : A \in K \setminus G\}) \subset U.$$

Since $H = X \setminus K$ is finite, the basic set

$$V = \{X\} \cup (X \setminus (H \cup G) \cup (\cup\{A \setminus F(A) : A \in X \setminus (H \cup G)\}$$

satisfies the conclusion of the lemma.

Lemma 6.4. *If $X \in \mathcal{A}_1$ and $\mathcal{B} \subset \mathcal{A}_1 \setminus \{X\}$ and $|\mathcal{B}| < \mathfrak{b}$, then there exists an open neighborhood U of X such that $cl(U) \cap \mathcal{B} = \emptyset$.*

Proof. For every $Y \in \mathcal{B}$ let $N(Y) \in \omega$ and $f(Y) \in {}^\omega\omega$ be such that $U(Y, N(Y), f(Y)) \cap U(X, 0, f(Y)) = \emptyset$. Let f be an upper bound (mod finite order) of $\{f(Y) : Y \in \mathcal{B}\}$. Put $U = U(X, 0, f)$. We show that $cl(U) \cap \mathcal{B} = \emptyset$. Fix $Y \in \mathcal{B}$. There exists $M \in \omega$ such that for all $i > M$, $f(Y)(i) \leq f(i)$. Define a function g so that $g \geq f(Y)$ and

$$g(i) > \max\{Y_i \cap (\cup_{j \leq M} X_j)\}$$

for all $i \geq N(Y)$. We show that $U(Y, N(Y), g) \cap U(X, 0, f) = \emptyset$. If not empty, there exists $i \geq N(Y), j \geq 0$ and $x \in \omega$ such that $x \in (Y_i \setminus g(i)) \cap (X_j \setminus f(j))$. If $j > M$ then $f(Y)(j) \leq f(j)$; hence

$$(Y_i \setminus g(i)) \cap (X_j \setminus f(j)) \subset (Y_i \setminus f(Y)(i)) \cap (X_j \setminus f(Y)(j)) = \emptyset.$$

Thus $j \leq M$. Hence $x \in Y_i \cap (\cup_{j \leq M} X_j)$; so $x < g(i)$ which contradicts $x \in Y_i \setminus g(i))$.

Proof of Theorem 1.7. Assume $\mathfrak{h} = \mathfrak{c}$. By Lemma 6.2, we may assume that $\mathcal{A}_0$ has the property that for any $X, Y \in [\mathcal{A}_0]^\omega$ with $|X \cap Y| < \omega$, there exists disjoint neighborhoods $U(X, Y)$ of X and $V(X, Y)$ of Y such that $cl(U(X, Y)) \cap cl(V(X, Y)) \cap \mathcal{A}_0 = \emptyset$. In other words, for every $A \in \mathcal{A}_0$ there exists a finite set $F(A)$ such that $A \setminus F(A) \cap U(X, Y) = \emptyset$ or $A \setminus F(A) \cap V(X, Y) = \emptyset$. List $[\mathcal{A}_0]^\omega = \{H_\alpha : \alpha < \mathfrak{c}\}$. We now construct the members of $\mathcal{A}_1$ by induction. Suppose we have constructed $X_\alpha \in [\mathcal{A}_0]^\omega$ for all $\alpha < \gamma$, where $\gamma < \mathfrak{c}$ such that

(i) if $\beta < \alpha$ then $X_\alpha \cap X_\beta$ is finite,

(ii) there exists $\beta \leq \alpha$ such that $X_\beta \cap H_\alpha$ is infinite,

(iii) for all pairs (β, τ) where $\beta < \tau < \alpha$, and all $\tau < \xi < \alpha$, $X_\xi \notin cl(U(X_\beta, X_\tau)) \cap cl(V(X_\beta, X_\tau))$.

We construct X_γ as follows. First pick any $H \in [\mathcal{A}_0]^\omega$ such that

$$\{X_\alpha : \alpha < \gamma\} \cup \{H\}$$

is an almost disjoint family such that at least one member intersects H_γ in an infinite set. This is possible because $\{X_\alpha : \alpha < \gamma\}$ is not maximal (since $\mathfrak{a} = \mathfrak{c}$). For $\alpha < \beta < \gamma$, and open set W we say that "W *decides* (α, β)" provided $W \cap U(X_\alpha, X_\beta) = \emptyset$ or $W \cap V(X_\alpha, X_\beta) = \emptyset$. Define

$$D(\alpha, \beta) = \{K \in [H]^\omega : \text{ some open neighborhood } W \text{ of } K \text{ decides } (\alpha, \beta)\}.$$

We show that $D(\alpha,\beta)$ is dense and open in $[H]^\omega$. Given $E \in [H]^\omega$, let $E' = \{A \in E : |A \cap U(X_\alpha, X_\beta)| < \omega\}$ if this set is infinite, and otherwise, let $E' = \{A \in E : |A \cap V(X_\alpha, X_\beta)| < \omega\}$. By the property on $\mathcal{A}_0$, E' is infinite. Clearly $E' \in D(\alpha,\beta)$. That $D(\alpha,\beta)$ is open follows from Lemma 6.3. Pick any

$$X_\gamma \in \cap\{D(\alpha,\beta) : \alpha < \beta < \gamma\}.$$

This is possible since the intersection of less that $\mathfrak{h}$ dense open set is non-empty. Since $X_\gamma \in D(\alpha,\beta)$ for all $\alpha < \beta < \gamma$, (iii) holds for X_γ. This completes the construction of $\mathcal{A}_1 = \{X_\alpha : \alpha < \mathfrak{c}\}$.

It follows from (i) and (ii) that $\mathcal{A}_1$ is a maximal almost disjoint family on $[\mathcal{A}_0]^\omega$. We show that $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is Urysohn. Since points in $\psi(\mathcal{A}_0)$ have local bases consisting of clopen sets in $\psi(\mathcal{A}_0, \mathcal{A}_1)$, it suffices to find a Urysohn separation for two points $X, Y \in \mathcal{A}_1$. By lemma 6.2 there exist disjoint neighborhoods $U = U(X,Y)$ of X and $V = V(X,Y)$ of Y such that $cl(U) \cap cl(V) \cap \mathcal{A}_0 = \emptyset$. By (iii), $|cl(U) \cap cl(V) \cap \mathcal{A}_1| < \mathfrak{c} = \mathfrak{b}$. By Lemma 6.4, there exists a neighborhood U' of X such that $cl(U') \cap (cl(U) \cap cl(V) \cap \mathcal{A}_1) = \emptyset$. Put $U(X) = U' \cap U$. Then $U(X)$ is a neighborhood of X and V a neighborhood of Y such that $cl(U(X)) \cap cl(V) = \emptyset$.

7. Proof of Theorem 1.6

Let $\mathcal{A}_0$ be maximal. Pick disjoint $X, Y \in [\mathcal{A}_0]^\omega$. Let $\{f_\alpha : \alpha \in \mathfrak{d}\}$ be dominating in $[\omega]^\omega$. By induction on $\mathfrak{d}$ we construct for $i < \omega$, $A(\alpha,i), B(\alpha,i) \in \mathcal{A}_0 \setminus (X \cup Y)$ such that

(a) $|A(\alpha,i) \cap (X \uparrow f_\alpha)| = |B(\alpha,i) \cap (Y \uparrow f_\alpha)| = \omega$
(b) $A(\alpha,i), B(\alpha,i) \notin \{A(\beta,j), B(\beta,j) : \beta < \alpha, j < \omega\} \cup \{A(\alpha,j), B(\alpha,j) : j < i\}$

Suppose we have constructed $A(\alpha,i), B(\alpha,i)$ for $\alpha < \gamma$ and $i < \omega$. We construct step γ.

Put

$$\mathcal{F} = \{A(\alpha,i), B(\alpha,i) : \alpha < \gamma, i < \omega\} \cup X \cup Y.$$

Since $\mathcal{F} \subset \mathcal{A}_0$, $\mathcal{F}$ is almost disjoint, and since $|\mathcal{F}| < \mathfrak{d}$, the set of those $F \in \mathcal{F}$ such that $|F \cap (X \uparrow f_\gamma)| = \omega$ is not maximal because $\mathfrak{d} \leq \mathfrak{a}$. Thus there exists an infinite $H \subset (X \uparrow f_\gamma)$ such that $F \cap H$ is finite for all $F \in \mathcal{F}$. Since $\mathcal{A}_0$ is maximal, there exists $A \in \mathcal{A}_0$ such that $A \cap H$ is infinite. Thus $A \notin \mathcal{F}$. We put $A(\gamma,0) = A$, and $\mathcal{F}_0 = \mathcal{F}$. Next we put $\mathcal{F}_1 = \mathcal{F}_0 \cup \{A(\gamma,0)\}$, and repeat the preceding argument to construct $A(\gamma,1) \in \mathcal{A}_0 \setminus (\mathcal{F} \cup \{A(\gamma,0)\})$ satisfying (a). We now repeat the preceding process countably many times to construct $A(\gamma,i)$ for $i < \omega$ satisfying (a), (b). In a similar manner, starting with $\mathcal{F} = \cup\{\mathcal{F}_i : i < \omega\}$ we construct $B(\gamma,i)$ for $i < \omega$. This completes the induction. For $\alpha < \mathfrak{d}$ we define $X_\alpha = \{A(\alpha,i), B(\alpha,i) : i < \omega\}$. By (b), $\{X_\alpha : \alpha < \mathfrak{d}\} \subset [\mathcal{A}_0]^\omega$ is a pairwise disjoint family. We claim that if $\mathcal{A}_1 \supset \{X_\alpha : \alpha < \mathfrak{d}\}$ (maximal or not) then $\psi(\mathcal{A}_0, \mathcal{A}_1)$ is not Urysohn. To see this consider two neighborhoods, $U(X,N,f)$ and $U(Y,N,f)$. Pick $\alpha < \mathfrak{d}$ such that $f \leq^* f_\alpha$. We show that X_α is in the closures of both neighborhoods. Consider an arbitrary neighborhood $U(X_\alpha, M, h)$. Let $X_\alpha = \{(X_\alpha)_i : i \in \omega\}$ be the preassigned listing of X_α. Pick $i > M$ such that $A(\alpha,i), B(\alpha,i) \in X_\alpha \setminus M$. Then there exists $j,k \in \omega$ such that $A(\alpha,i) = (X_\alpha)_j, B(\alpha,i) = (X_\alpha)_k$. Put $A = (X_\alpha)_j$ and $B = (X_\alpha)_k$. Then by (a)

$$|(A \setminus h(j)) \cap (X \uparrow f_\alpha)| = |(B \setminus h(k)) \cap (Y \uparrow f_\alpha)| = \omega.$$

Hence

$$|(A \setminus h(j)) \cap ((X \setminus N) \uparrow f_\alpha)| = |(B \setminus h(k)) \cap ((Y \setminus N) \uparrow f_\alpha)| = \omega.$$

Since $f \leq^* f_\alpha$, we have that $U(X_\alpha, M, h) \cap U(X, N, f) \neq \emptyset$ and $U(X_\alpha, M, h) \cap U(Y, N, f) \neq \emptyset$. Thus

$$X_\alpha \in \overline{U(X, N, f)} \cap \overline{U(Y, N, g)}.$$

To complete the proof we take $\mathcal{A}_1 \supset \{X_\alpha : \alpha < \mathfrak{d}\}$ to be maximal.

In 1992, James Baumgartner sent us a proof showing that the conclusion of Theorem 1.6 holds in forcing models obtained by adding random reals to a model of CH. Some years later we proved Theorem 1.6 from which his result follows since $\mathfrak{d} = \omega_1$ in such models. The idea of Baumgartner's proof serves to prove the following result.

Theorem 7.1. *Assume $\mathfrak{d} < \mathfrak{r}$. For every maximal $\mathcal{A}_0$ and any $Z \in [\mathcal{A}_0]^\omega$, there exists a partition of Z into two infinite sets X, Y such that $\psi(\mathcal{A}_0, \{X, Y\})$ is not Urysohn.*

Proof. Let $\mathcal{A}_0 \subset [\omega]^\omega$ be maximal, and $Z = \{A_i : i \in \omega\} \in [\mathcal{A}_0]^\omega$. Let $D \subset {}^\omega\omega$ be a dominating family of increasing functions with $|D| = \mathfrak{d}$. For each $f \in D$, by maximality pick $A_f \in \mathcal{A}_0$ such that $A_f \cap (Z \uparrow f)$ is infinite. Let $d_f = \{i \in \omega : A_f \cap (A_i \setminus f(i))\} \neq \emptyset\}$. Then $\{d_f : f \in D\}$ is not a refining family; so there exists $H \in [\omega]^\omega$such that for all $f \in D$, both $H \cap d_f$ and $(\omega \setminus H) \cap d_f$ are infinite. Put $X = \{A_i : i \in H\}$ and $Y = \{A_i : i \in \omega \setminus H\}$. To see that X, Y are as required, consider neighborhoods $U(X, N, g)$ and $U(Y, N, g)$ where $N \in \omega$ and $g \in {}^\omega\omega$. There exists $f \in D$ such that $g \leq^* f$. Pick M so that if $i \geq M$, then $g(i) \leq f(i)$. Now d_f has infinite intersection with both H and $\omega \setminus H$. We show that the point A_f is in the closure of both $U(Y, N, g)$ and $U(X, N, g)$. Since $d_f \cap H$ is infinite, we may pick $i > N, M$ such that $i \in d_f \cap H$ and $g(i) \leq f(i)$. Hence $A_f \cap A_i \setminus g(i) \neq \emptyset$. Thus $A_f \in cl(U(X, N, g))$. Similarly, $A_f \in cl(U(Y, N, g))$. This completes the proof.

Corollary 7.2. *Assume $\mathfrak{d} < \mathfrak{r}$. No maximal $\mathcal{A}_0$ satisfies the conclusion of Lemma 6.2. Thus whether there exists a maximal $\mathcal{A}_0$ satisfying the conclusion of Lemma 6.2 is consistent with and independent of ZFC.*

We thank James Baumgartner for his discussion of $\psi(\mathcal{A}_0, \mathcal{A}_1)$ in the random real model, and Alan Dow for the observation that our earlier proof of Theorem 1.7, using $\mathfrak{p} = \mathfrak{c}$, works using the weaker hypothesis $\mathfrak{h} = \mathfrak{c}$.

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